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Crisis-induced intermittency and Melnikov scale factor

Marek Franaszek [1], Emil Simiu

Building and Fire Research Laboratory, National Institute of Standards and Technology, Building 226, Gaithersburg, MD 20899, USA

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Abstract

We study the post-critical behavior of a perturbed bistable Hamiltonian system to which the Melnikov approach is applicable under the assumption that the perturbation is asymptotically small. We examine the case of perturbations that are sufficiently large to cause chaotic transport between phase space regions associated with the system's potential wells. The main results are: (1) a small additional harmonic excitation can cause substantial changes in the system's mean residence time, and (2) the dependence of the magnitude of these changes on the additional excitation's frequency is similar to the dependence on frequency of the system's Melnikov scale factor. We discuss the relevance of these results to the design of efficient, Melnikov-based open loop controls aimed at increasing the mean residence time for the stochastically excited counterpart of the system.

Crisis-induced intermittency may be observed in a dynamical system when a control parameter λ has a critical value λ_0. For $\lambda < \lambda_0$ two coexisting chaotic attractors exist, while for $\lambda > \lambda_0$ the two attractors are merged into one large attractor. Following a crisis, an endless sequence of alternating excursions to the regions previously occupied by the two attractors is characterized by the probability distribution

$$P(t) = P_0\,e^{-t/\tau}, \qquad (1)$$

where $P(t)$ denotes the probability of observing an excursion of duration t or longer, and τ is the mean residence time inside a particular region [1,2]. The time τ depends on the control parameter $\lambda > \lambda_0$,

$$\tau(\lambda) \sim (\lambda - \lambda_0)^{-\delta}, \qquad (2)$$

where δ may be expressed as a function of the eigenvalues associated with the unstable periodic orbit responsible for the crisis [1]. However, these useful relations do not predict the parameter values for which one can expect post-critical behavior.

For a wide class of nonlinear systems a necessary condition for crisis occurrence may be formulated in terms of the Melnikov function [3–5]. One example on which we focus for definiteness is the Duffing–Holmes equation

$$\ddot{x}(t) = F(x, \dot{x}, t), \qquad (3)$$

where

$$F(x, \dot{x}, t) = 0.5x - 0.5x^3 - \gamma\dot{x} + \lambda\sin(\Omega t). \qquad (4)$$

In the absence of perturbation ($\gamma = \lambda = 0$) the system has two homoclinic loops originating from the point $(x, \dot{x}) = (0, 0)$. In accordance with Melnikov theory, an endless sequence of jumps between the regions of

[1] Permanent address: Institute of Physics, Cracow Pedagogical University, Podchorążych 2, Cracow, Poland.

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$x(t) > 0$ and $x(t) < 0$ – i.e., steady-state chaotic transport across a pseudoseparatrix [5] – is possible only if the Melnikov function

$$M(t_0,t) = -\tfrac{1}{3}\sqrt{8}\gamma+2\pi\lambda S(\Omega)\sin[\Omega(t_0+t)] \quad (5)$$

has simple zeros; in Eq. (5) $S(\Omega)$ is the Melnikov scale factor (i.e., the modulus of the Fourier transform of $\dot{x}(-t)$, where $\dot{x}(-t)$ is the ordinate of the homoclinic orbit in the phase plane $[x(t),\dot{x}(t)]$ [6]), and t_0 is related to the initial phase of the sinusoidal forcing. The derivation of this necessary condition is based on the assumption that the system's perturbation is asymptotically small ($\gamma \to 0$ and $\lambda \to 0$); strictly speaking, it is valid only if this assumption holds. However, there is numerical evidence that Melnikov theory is helpful in the search for chaos even for relatively large perturbations [7,8]. In this Letter we show that, for relatively large perturbations, Melnikov theory also has a useful role in describing the effect of additional small perturbations on the mean residence time.

We consider the case $\lambda = \lambda_0 + \Delta\lambda$, $\Delta\lambda > 0$ and $\Delta\lambda/\lambda_0 \ll 1$, i.e., λ is close to and larger than its critical value. The mean residence time τ is therefore large, see Eq. (2). We now add to the system a second harmonic perturbation of amplitude A and frequency ω. The equation of motion is then

$$\ddot{x}(t) = F(x,\dot{x},t) + A\sin(\omega t), \quad (6)$$

while the corresponding Melnikov function is given by

$$M(t_{01},t_{02},t) = -\tfrac{1}{3}\sqrt{8}\gamma+2\pi\{\lambda S(\Omega)\sin[\Omega(t+t_{01})] + AS(\omega)\sin[\omega(t+t_{02})]\}. \quad (7)$$

We integrate Eq. (5) numerically and obtain the mean residence time τ as a function of ω for fixed γ, Ω and $\Delta\lambda$, and various values of the amplitude A.

The main result of this Letter is that the addition of a second small harmonic perturbation may change τ drastically, and that the dependence of this change on the frequency ω is similar to the dependence on ω of the Melnikov scale factor. For example, additional forcing with amplitude $A \approx 0.02\lambda$ may reduce the mean residence time by a factor of two or more for sufficiently small $\Delta\lambda$. Later in this work we discuss how this result may help to perform an efficient control of the stochastically excited nonlinear oscillator.

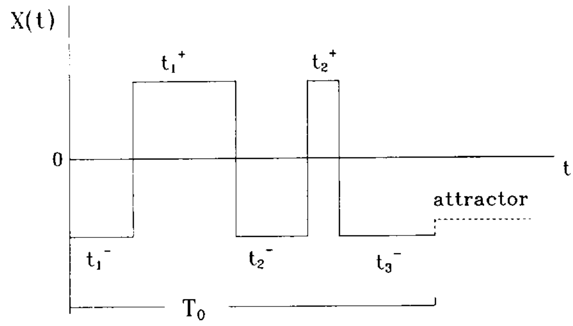

Fig. 1. Schematic evolution of chaotic transients observed before crisis. For another initial point a trajectory may approach the second coexisting attractor with $x(t) > 0$.

Before presenting the results obtained for $\lambda > \lambda_0$ and $A > 0$, we wish to describe the behavior of the oscillator excited by just one harmonic term for $\lambda < \lambda_0$ and $A = 0$. A study of this case is essential for understanding crisis-induced intermittency observed in the system. We let $\Omega = 0.89$, $\gamma = 0.045$. For these parameters the crisis occurs for $\lambda_0 = 0.114358\ldots$. For λ_1 close to and smaller than the critical value (e.g., $\lambda_1 = 0.1142$) the system has three coexisting attractors: one periodic and two chaotic. Of the two chaotic attractors, one is confined to the region of exclusively positive, and the other to the region of exclusively negative values of $x(t)$. Starting from an initial point x_0 belonging to the basin of attraction of one of the chaotic attractors, a trajectory will eventually settle on that attractor. Before this occurs one may observe a long chaotic transient of duration T_0. During this transient a trajectory alternates between the regions of positive and negative $x(t)$, see Fig. 1. The duration of a particular transient, T_0 and a particular sequence of time intervals $t_1^-, t_2^-, \ldots$, depend strongly on x_0. Starting from many initial points x_0 we can determine the distributions $P(T_0)$ and $P(t)$. (Owing to the symmetry of the potential, $P(t^+) = P(t^-)$.) The results of the simulations are shown in Fig. 2. Note that the distribution $P(T_0)$ has a simple exponential form with very long mean lifetime $\tau_0 = 1454$. On the other hand, the distribution of the residence time $P(t)$ is a superposition of two different exponential decays: a faster decay characterized by $\tau_1 = 19.1$, and a slower decay with $\tau_2 = 54.8$. Here, τ_1 and τ_2 denote the inverse slopes of straight lines, the two lines being fitted separately in two adjacent ranges of t.

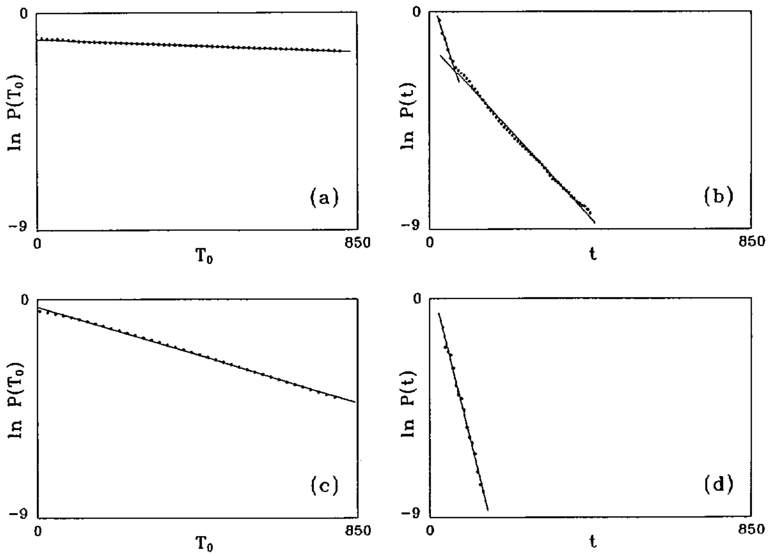

Fig. 2. (a) Distribution of transient times T_0 for $\lambda = \lambda_1$; (b) distribution of residence times t for $\lambda = \lambda_1$; corresponding distributions obtained for $\lambda = \lambda_2$ are shown in (c) and (d). Each distribution was generated by integrating Eq. (3) for $N_0 = 2 \times 10^5$ uniformly distributed initial points $(x_0, \dot{x}_0)$.

Transient chaos and exponential distributions of lifetimes are usually caused by the existence of a nonattracting chaotic set. A well known example of such a set is a fractal basin boundary or a strange repellor which is a remnant of a chaotic attractor destroyed in a crisis [9]. In our case, for $\lambda_1 < \lambda_0$ the repellors are not yet born and the only nonattracting sets are three different basin boundaries (recall that for $\lambda = \lambda_1$ there are three coexisting attractors). A distribution such as that shown in Fig. 2b must be associated with the complicated structure of three coexisting nonattracting sets. To lend support to this hypothesis we repeat the numerical simulations for a smaller amplitude $\lambda_2 = 0.08$. For this value of the control parameter there are only two coexisting periodic attractors (they are precursors of the two chaotic attractors that exist for λ_1). Thus, for $\lambda = \lambda_2$ there is only one nonattracting chaotic set, i.e. a boundary separating two basins. The distribution of the residence time, $P(t)$, is shown in Fig. 2d and it has a simple exponential form with mean residence time $\tau_1 = 14.8$ (compare with τ_1 in Fig. 2b). Note that the distribution $P(T_0)$ shown in Fig. 2c is again a simple exponential, but the mean lifetime $\tau_0 = 199$ is nearly ten times smaller than the corresponding τ_0 in Fig. 2a.

Keeping these facts in mind we are now ready to analyze post-critical behavior of the Duffing–Holmes oscillator. When λ becomes larger than λ_0 two chaotic attractors are destroyed and replaced by two repellors characterized by a finite mean lifetime. Thus, together with the three nonattracting sets that existed before the crisis, we have now five coexisting repulsing sets and the behavior of the system is more complicated. Such behavior may be viewed as an example of multitransient chaos [10]. In Fig. 3a the resulting distribution of the residence time, $P(t)$, is shown for $\lambda = \lambda_0 + \Delta\lambda$ and $\Delta\lambda = 2 \times 10^{-4}$. Three independent straight line fits yield the value of $\tau_1 = 13.3$, $\tau_2 = 54.4$, and $\tau_3 = 208.3$. The last straight line slope depends sensitively on $\Delta\lambda$, see Eq. (2), while the first two are nearly independent of λ and are very close to the values obtained from the data shown in Figs. 2b and 2d. We remark that after the crisis we have only two coexisting attractors: the large chaotic attractor and the simple periodic attractor. Thus, the number of basin boundaries is reduced by the crisis from three to one. However, the nonattracting sets existing for $\lambda < \lambda_0$ do not vanish. They remain nearly unaffected by the crisis and reveal their presence either as a part of a new basin boundary or as a part of a new large chaotic attractor [11].

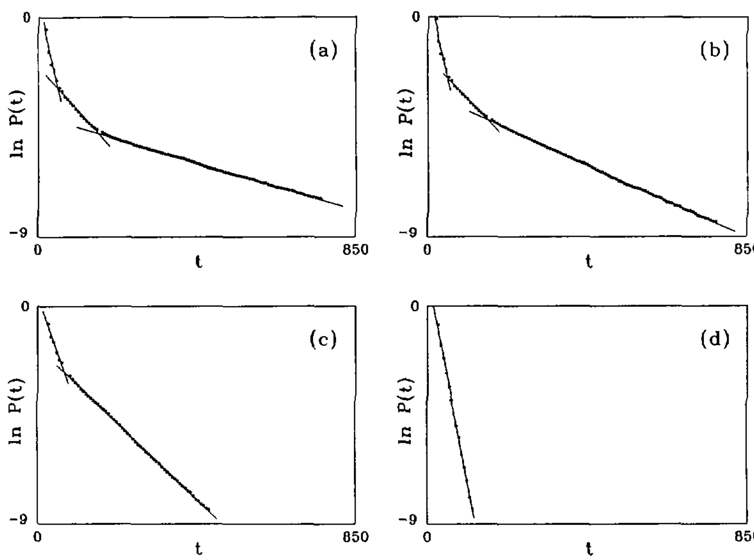

Fig. 3. Distribution of residence times t obtained for $\lambda = \lambda_0 + \Delta\lambda$ from one long stationary trajectory. (a) Without additional perturbation, i.e. $A = 0$; with added second harmonic perturbation of amplitude A and frequency ω: (b) $A = 2.5\Delta\lambda$, $\omega = 0.8455$; (c) $A = 20\Delta\lambda$, $\omega = 1.2905$; (d) $A = 160\Delta\lambda$, $\omega = 0.6230$. In each case a length of trajectory was $N = 1.5 \times 10^6 \Delta T$ where $\Delta T = 2\pi/\Omega$ is the period of harmonic excitation with amplitude λ.

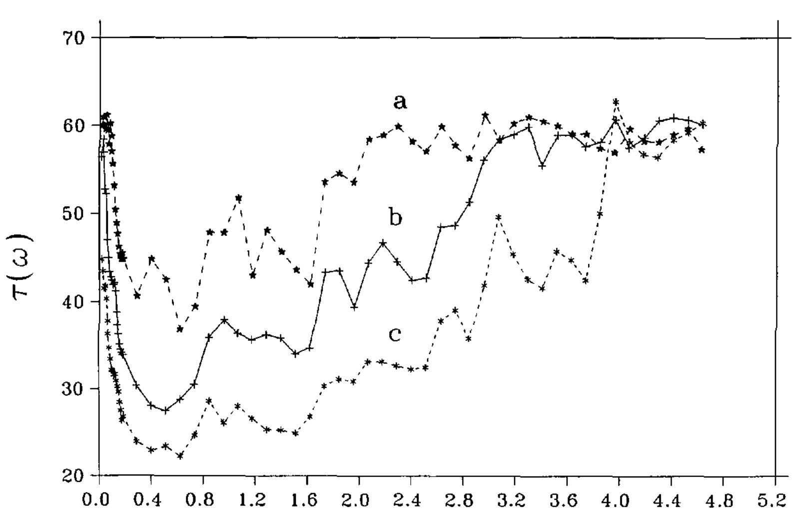

Fig. 4. Dependence of mean residence time τ on frequency ω for three different amplitudes A: (a) $A = 2.5\Delta\lambda$; (b) $A = 20\Delta\lambda$; (c) $A = 160\Delta\lambda$.

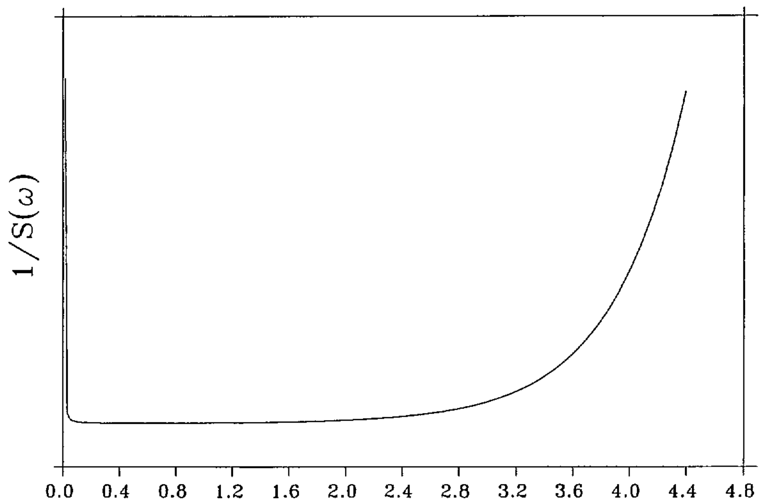

Fig. 5. Inverse of Melnikov scale factor $S(\omega)$, see Eqs. (3), (4) with $\gamma = \lambda = 0$. $S(\omega)$ approaches zero as $\omega \to 0$ or $\omega \to \infty$.

Now, we switch on a second harmonic excitation of amplitude A and frequency ω. We wish to investigate the influence of this additional term on the crisis-induced intermittency observed for $\lambda = \lambda_0 + \Delta\lambda$. We show in Figs. 3b, 3c and 3d examples of residence time distribution $P(t)$ obtained for different values of A and ω. It is clear that the effectiveness of this new perturbation depends strongly on both amplitude A and frequency ω. For example, in Fig. 3b the additional perturbation caused the slowest decay to be modified slightly while in Fig. 3d it destroyed the two slower decays and only the fastest decay survived. In order to make our investigation more systematic, we keep the amplitude A fixed and change the frequency ω within a certain interval. For each ω we obtain the distribution $P(t)$, perform appropriate straight line fits, and calculate the mean residence time $\tau(\omega)$. In Fig. 4 the resulting dependence $\tau(\omega)$ is shown for three different values of the amplitude A. For any given amplitude A the mean residence time and the inverse of the Melnikov scale factor exhibit remarkably similar dependences on ω, see Fig. 5. Moreover, as shown by Fig. 4, the mean residence time decreases monotonically (i.e., the system becomes increasingly chaotic) as the amplitude A increases.

We found that for fixed A the shape of the distribution $P(t)$ changes in accordance with a well-defined pattern. For small ω all three types of decay are present. As the frequency ω gradually increases, the slowest decay is destroyed first, and the intermediate decay is destroyed next. Further increasing ω brings the intermediate decay back into existence, and then the slowest decay is also restored. We never observed a deviation from this regular pattern.

To summarize our observations: given a system with relatively large perturbation, (1) the addition of a small harmonic excitation can significantly influence the system's mean residence time, and (2) the dependence of that influence on the excitation's frequency is similar to the dependence on frequency of the Melnikov scale factor.

These observations have potential application to the control of nonlinear stochastically excited systems to which Melnikov theory is applicable – a wide class of systems is discussed in some detail elsewhere [6,12]. The objective of the control system would be to decrease substantially the mean escape rate from a "safe" region of phase space. Our results suggest that such a result could be achieved by an open loop control system with a relatively small control force that would counteract the stochastic excitation (i.e., whose Fourier transform components would reduce their counterparts in the Fourier transform of the excitation). To keep the power requirements as low as possible, the control force would be obtained from the stochastic excitation via a filter designed to reduce or eliminate ineffective Fourier transform components of the control, that is, components corresponding to frequencies for which the Melnikov scale factor is small. The results presented in this Letter are a preliminary step in a study of Melnikov-based controls currently conducted by the authors.

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